THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 3

Date: 26 September, 2024

Field Axioms of real number:

A1. $a + b \in \mathbb{R}$ if $a, b \in \mathbb{R}$;

A2. $a + b = b + a$ if $a, b \in \mathbb{R}$;

A3. $a + (b + c) = (a + b) + c \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$;

A4. There exists $0 \in \mathbb{R}$ such that $a + 0 = a$ for all $a \in \mathbb{R}$;

A5. For any $a \in \mathbb{R}$, there is $b \in \mathbb{R}$ such that $a + b = 0$;

M1. $a \cdot b \in \mathbb{R}$ if $a, b \in \mathbb{R}$;

M2. $a \cdot b = b \cdot a$ if $a, b \in \mathbb{R}$;

M3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$;

M4. There exists $1 \in \mathbb{R} \setminus \{0\}$ such that $a \cdot 1 = a$ for all $a \in \mathbb{R}$;

M5. For any $a \in \mathbb{R} \setminus \{0\}$, there is $b \in \mathbb{R}$ such that $a \cdot b = 1$;

D. $a \cdot (b + c) = a \cdot b + a \cdot c$ if $a, b, c \in \mathbb{R}$.

1. (a) State the completeness of $\mathbb{R}$;
   (b) Using the axioms (and point out which axiom is used at each step), show that
      i. $(-a) \cdot (-b) = a \cdot b$;
      ii. $1/(-a) = -(1/a)$ if $a \neq 0$.

Pf: 1) a) Suppose $S \subseteq \mathbb{R}$ is a nonempty set in $\mathbb{R}$ and is bounded from above. Then $\sup S$ exists in $\mathbb{R}$.

b) i) $0 = a \cdot 0$ for all $a \in \mathbb{R}$:

$$\begin{aligned} 0 &= a\cdot 0 + (-a\cdot 0) && (A5) \\ &= a\cdot(0+0) + (-a\cdot 0) && (A4) \\ &= a\cdot 0 + \cancel{a\cdot 0} + \cancel{(-a\cdot 0)} && (D) \\ &= a\cdot 0 && (A5). \end{aligned}$$

Strictly speaking, also need to show uniqueness of inverses for the arguments below.

$a\cdot(-1) = -a$ for all $a \in \mathbb{R}$:

$$\begin{aligned} 0 &= a\cdot 0 && (\text{proved}) \\ &= a\cdot(1+(-1)) && (A5) \end{aligned}$$

$= a\cdot 1 + a\cdot(-1)$ (D)

$= a + a\cdot(-1)$ (M4).

$\Rightarrow$ by uniqueness of additive inverse, $a\cdot(-1) = -a$.

Also need: $(-1)^2 = 1$.

$(-1)^2 = (-1)^2 + 0$ (A4)

$= (-1)^2 + (-1) + 1$ (A5)

$= (-1)\cdot(-1) + (-1) + 1$ (def'n of square).

$= (-1)\cdot(-1) + (-1)\cdot 1 + 1$ (M4)

$= (-1)\cdot((-1) + 1) + 1$ (D)

$= (-1)\cdot 0 + 1$ (A5)

$= 0 + 1$ (proved)

$= 1$. (A4).

$$(-a)\cdot(-b) = a\cdot(-1)\cdot b\cdot(-1) \qquad \text{(proved)}$$

$$= (-1)^2 \cdot a \cdot b \qquad \text{(M2)}$$

$$= 1 \cdot a \cdot b \qquad \text{(proved)}$$

$$= a \cdot b.$$

iv) $$\frac{1}{-a} - \left(-\frac{1}{a}\right) = \frac{1}{a\cdot(-1)} - \left(\frac{1}{a}\cdot(-1)\right) \qquad \text{(proved)}$$

$$= \frac{1}{a}\cdot\left(\frac{1}{-1} - (-1)\right) \qquad \text{(D)}$$

$$= \frac{1}{a}\cdot 0 \qquad \text{(A5)}$$

$$= 0. \qquad \text{(proved)}.$$

2. Suppose S is a bounded non-empty subset in $\mathbb{R}$.

   (a) Show that $\inf(-S) = -\sup S$ where $-S = \{-x : x \in \S\}$.

   (b) Show that $\sup S_0 \leq \sup S$ if S_0 is an non-empty subset of S.

Pf: a) Since S is bounded, $-S$ is also bounded.

Since S is nonempty, $-S$ is also nonempty.

Hence by completeness axiom, $\inf(-S)$ exists.

Let $u := \sup S$. WTS $-u$ is a lower bdd. of $-S$:

Since $u = \sup S$, $\forall s \in S$, we have $s \leq u$.

Multiplying by -1, we have $-u \leq \underset{-S}{\underset{\cap}{-s}}$ for all $s \in S$.

So $-u$ is a lower bdd. of $-S$.

Greatest lower bdd. property: Let v be a lower bdd. of $-S$.

WTS: $v \leq -u$. Since v is a lower bdd. of $-S$, we have

$v \leq -s \quad \forall s \in S$.

So multiplying by -1, we have $s \leq -v$ for all $s \in S$.

So $-v$ is an u.b. of S. Then since $u = \sup S$, we have $u \leq -v$. Then multiplying again by -1 gives

$v \leq -u$.

Hence, $-\sup S = -u = \inf(-S)$.

b) S_0 is nonempty by assumption.

$S_0 \subseteq S$ and S is bounded, so S_0 is bounded:

bounded above: S is bounded above, so $\exists M \in \mathbb{R}$ s.t.

$s \leq M$ for all $s \in S$.

Let $s_0 \in S_0 \subseteq S$. So in particular $s_0 \leq M$. Since s_0 was arbitrary, S_0 is bounded above by M.

By completeness axiom, $\sup S_0$ exists in $\mathbb{R}$.

WTS $\sup S_0 \leq \sup S$: Sps for the sake of contradiction that $\sup S < \sup S_0$. Since $\sup S_0$ is l.u.b. of S_0, this means that $\sup S$ is <u>not</u> an u.b. of S_0. So, we can find an $s_0 \in S_0$ s.t. $s_0 > \sup S$. But then since $s_0 \in S$, this contradicts the fact that $\sup S$ is an u.b. of S. ✓

3. By considering $\mathcal{A} = \{x \in \mathbb{R} : x^2 + x < 3\}$, show that there exists $u \in \mathbb{R}$ such that $u > 0$ and $u^2 + u = 3$.

Pf: WTS $u = \sup \mathcal{A}$ will satisfy $u > 0$, and $u^2 + u = 3$.

First show $u = \sup \mathcal{A}$ exists.

nonempty: $1^2 + 1 = 2 < 3$, so $1 \in \mathcal{A}$ and $\mathcal{A}$ is nonempty.

bounded above: $2^2 + 2 = 6 > 3$. so $\mathcal{A}$ is bounded from above by 2.

So by completeness axiom, $u = \sup \mathcal{A}$ exists in $\mathbb{R}$

2nd step: $u > 0$: Since $1 \in \mathcal{A}$ and u is an u.b. of $\mathcal{A}$, we have $u \geq 1 > 0$.

3rd step: $u^2 + u = 3$.

First sps $u^2 + u > 3 \Rightarrow u^2 + u - 3 > 0$.

Contradiction we want: $u - \frac{1}{m}$ is an u.b. of $\mathcal{A}$.

$$\left(u - \frac{1}{m}\right)^2 + \left(u - \frac{1}{m}\right) > 3.$$

$$\begin{aligned}\left(u - \frac{1}{m}\right)^2 + \left(u - \frac{1}{m}\right) &= u^2 - \frac{2u}{m} + \frac{1}{m^2} + u - \frac{1}{m} \\ &> u^2 + u - \frac{2u}{m} - \frac{1}{m} \\ &= u^2 + u - \frac{1}{m}(2u+1)\end{aligned}$$

So pick $m \in \mathbb{N}$ s.t. $\frac{1}{m} < \frac{u^2 + u - 3}{2u+1}$.

Such an $m \in \mathbb{N}$ exists b/c $u^2 + u - 3 > 0$, $u > 0 \Rightarrow 2u + 1 > 0$

so $\frac{u^2+u-3}{2u+1} > 0$. and we can pick such an m by A.P.

So with this m chosen, we obtain

$$\left(u-\frac{1}{m}\right)^2+\left(u-\frac{1}{m}\right) > u^2+u-\frac{u^2+u-3}{\cancel{(2u+1)}}\cancel{(2u+1)}$$
$$=3.$$

So $u-\frac{1}{m}$ is an u.b. of A, a contradiction.

Now sps $u^2+u<3$.

$$\left(u+\frac{1}{n}\right)^2+\left(u+\frac{1}{n}\right)=u^2+\frac{2u}{n}+\frac{1}{n^2}+u+\frac{1}{n}$$
$$\leq u^2+u+\frac{1}{n}(2u+2)$$

Then since $u>0$, $2u+2>0$, $3-u^2-u>0$, by A.P. $\exists n\in\mathbb{N}$ s.t.

$$\frac{1}{n}<\frac{3-u^2-u}{2u+2}$$ and so

$$\left(u+\frac{1}{n}\right)^2+\left(u+\frac{1}{n}\right) \leq u^2+u+\left(\frac{3-u^2-u}{\cancel{2u+2}}\right)\cancel{(2u+2)}$$

$$=3.$$

So $u+\frac{1}{n} \in A$, but clearly $u+\frac{1}{n} > u=\sup A$, a contradiction.

So by trichotomy property, $u^2+u=3$.